Homework Logistics

- HW1 is due this Friday 09/02.
- Login to Gradescope *TODAY* to see if you have access to CS 70. If you still do not have access, read https://piazza.com/class/irwxcmgdofp2uz?cid=6.
- Homework must be submitted electronically to Gradescope as pdf but it may be prepared by hand, in LaTeX, or using Microsoft Word. Make sure the uploaded PDF is readable!!!

Lecture Outline

Strengthening Induction Hypothesis.

Strong Induction

Well ordered principle.

Strengthening Induction Hypothesis.

Theorem: The sum of the first n odd numbers is a perfect square.

Theorem: The sum of the first n odd numbers is k^2.

kth odd number is $2k-1$ for $k \geq 1$.

Base Case: 1 (1st odd number) is 1^2.

Induction Hypothesis: Sum of first k odds is perfect square $a^2 = k^2$.

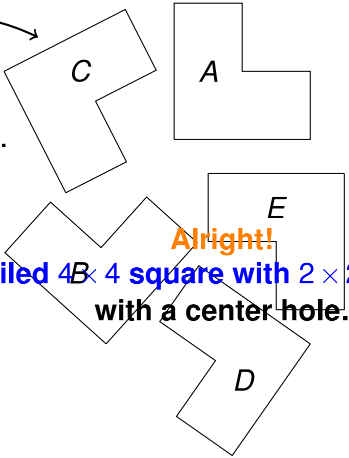
Induction Step: To prove that sum of first $k+1$ odds is $(k+1)^2$.

1. The $(k+1)$st odd number is $2(k+1)-1$ = $2k+1$.
2. Sum of the first $k+1$ odds is $a^2+2k+1 = k^2+2k+1$
3. $k^2+2k+1 = (k+1)^2$
   ... P(k+1)!

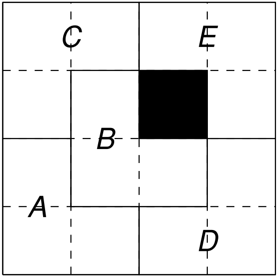
$\square$

Tiling Cory Hall Courtyard.

Use these L-tiles.



To Tile this 4×4 courtyard.



Alright!

Tiled 4×4 square with 2×2 L-tiles.

with a center hole.

Can we tile any $2^n \times 2^n$ with L-tiles (with a hole) for every n!

Hole have to be there? Maybe just one?

Theorem: Any tiling of $2^n \times 2^n$ square has to have one hole.

Proof: Each tile covers 3 squares. The remainder of 2^{2n} divided by 3 is 1.

Base case: true for $n = 0$. $2^0 = 1$

Ind Hyp: $n = k$. $2^{2k} = 3a + 1$ for integer a.

$$
\begin{aligned}
2^{2(k+1)} &= 2^{2k} * 2^2 \\
&= 4 * 2^{2k} \\
&= 4 * (3a + 1) \\
&= 12a + 3 + 1 \\
&= 3(4a + 1) + 1
\end{aligned}
$$

a integer $\implies$ $(4a + 1)$ is an integer. $\square$

Hole in center?

Theorem: Can tile the $2^n \times 2^n$ square to leave a hole adjacent to the center.

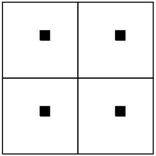
Proof:

Base case: A single tile works fine.
The hole is adjacent to the center of the 2×2 square.

Induction Hypothesis:
Any $2^n \times 2^n$ square can be tiled with a hole at the center.

2^{n+1}

2^{n+1}



2^n

2^n

What to do now???

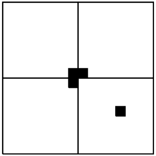
Hole can be anywhere!

Theorem: Can tile the $2^n \times 2^n$ to leave a hole adjacent *anywhere.*

Better theorem ... stronger induction hypothesis!

Base case: Sure. A tile is fine.
Flipping the orientation can leave hole anywhere.

Induction Hypothesis:
"Any $2^n \times 2^n$ square can be tiled with a hole **anywhere.**"
Consider $2^{n+1} \times 2^{n+1}$ square.



Use induction hypothesis in each.

Use L-tile and ... we are done. $\square$

Strong Induction: Example

Theorem: Every natural number $n > 1$ is either a prime or can be written as a product of primes.

Fact: A prime n has exactly 2 factors 1 and n.

Base Case: $n = 2$.

Induction Step:

$P(n) =$ "n is either a prime or a product of primes. "

Either $n+1$ is a prime or $n+1 = a \cdot b$ where $1 < a, b < n+1$.

$P(n)$ says nothing about a, b!

Strong Induction Principle: If $P(0)$ and

$$(\forall k \in N)((P(0) \wedge \ldots \wedge P(k)) \implies P(k+1)),$$

then $(\forall k \in N)(P(k))$.

$$P(0) \implies P(1) \implies P(2) \implies P(3) \implies \cdots$$

Strong induction hypothesis: "a and b are products of primes"

$\implies$ "$n+1 = a \cdot b =$ (factorization of a)(factorization of b)"

$n+1$ can be written as the product of the prime factors!

$\square$

Strong Induction is a form of (regular) Induction.

Let $Q(k) = P(0) \wedge P(1) \cdots P(k)$.

By the induction principle:
"If $Q(0)$, and $(\forall k \in N)(Q(k) \implies Q(k+1))$ then $(\forall k \in N)(Q(k))$"

Also, $Q(0) \equiv P(0)$, and $(\forall k \in N)(Q(k)) \equiv (\forall k \in N)(P(k))$

$$(\forall k \in N)(Q(k) \implies Q(k+1))$$
$$\equiv (\forall k \in N)((P(0) \cdots \wedge P(k)) \implies (P(0) \cdots P(k) \wedge P(k+1)))$$
$$\equiv (\forall k \in N)((P(0) \cdots \wedge P(k)) \implies P(k+1))$$

Strong Induction Principle: If $P(0)$ and

$$(\forall k \in N)((P(0) \wedge \ldots \wedge P(k)) \implies P(k+1)),$$

then $(\forall k \in N)(P(k))$.

Well Ordering Principle and Induction.

If $(\forall n)P(n)$ is not true, then $(\exists n)\neg P(n)$.

Consider smallest m, with $\neg P(m)$,

$P(m-1) \implies P(m)$ must be false (assuming $P(0)$ holds.)

This is a proof of the induction principle!
I.e.,

$$\neg(\forall n P(n)) \implies \big((\exists n)\neg(P(n-1) \implies P(n)\big).$$

(Contrapositive of Induction principle (assuming $P(0)$)

It assumes that there is a smallest m where $P(m)$ does not hold.

The **Well ordering principle** states that for any subset of the natural numbers there is a smallest element.

Horses of the same color...

Theorem: All horses have the same color.

Base Case: $P(1)$ - trivially true.
New Base Case: $P(2)$: there are two horses with same color.

Induction Hypothesis: $P(k)$ - Any k horses have the same color.

Induction step $P(k+1)$?

First k have same color by $P(k)$. $1, 2, 3, \ldots, k, k+1$

Second k have same color by $P(k)$. $1, 2, 3, \ldots, k, k+1$

A horse in the middle in common! $1, 2, 3, \ldots, k, k+1$

All k must have the same color. No horse in common! $1, 2, 3, \ldots, k, k+1$

How about $P(1) \Longrightarrow P(2)$?
Fix base case.
...Still doesn't work!!
(There are two horses is $\not\equiv$ For all two horses!!!)

Of course it doesn't work.

As we will see, it is more subtle to catch errors in proofs of correct theorems!!

Strong Induction and Recursion.

Thm: For every natural number $n \geq 12$, $n = 4x + 5y$.

Instead of proof, let's write some code!

```
def find-x-y(n):
   if (n==12) return (3,0)
   elif (n==13): return(2,1)
   elif (n==14): return(1,2)
   elif (n==15): return(0,3)
   else:
     (x,y) = find-x-y(n-4)
     return(x+1,y)
```

Base cases: P(12) , P(13) P(14) P(15). Yes.

Strong Induction step:
Recursive call is correct: $P(n-4) \implies P(n)$.

Slight differences: showed for all $n \geq 16$ that $\wedge_{i=4}^{n-1} P(i) \implies P(n)$.

Summary: principle of induction.

Today: More induction.

$(P(0) \wedge ((\forall k \in N)(P(k) \implies P(k+1)))) \implies (\forall n \in N)(P(n))$

Statement to prove: $P(n)$ for n starting from n_0
Base Case: Prove $P(n_0)$.
Ind. Step: Prove. For all values, $n \geq n_0$, $P(n) \implies P(n+1)$.
Statement is proven!

Strong Induction:
$(P(0) \wedge ((\forall n \leq k P(n)) \implies P(k+1))) \implies (\forall n \in N)(P(n))$

Also Today: strengthened induction hypothesis.

Strengthen theorem statement.
Sum of first n odds is n^2.
Hole anywhere.
Not same as strong induction.

Induction $\equiv$ Recursion.