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Original Problem Setting:

- Small town with n men and n women.
- Each woman has a ranked preference list of men.
- Each man has a ranked preference list of women.

How should they be matched?

What criteria to use?

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- Maximize number of first choices.

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- Minimize difference between preference ranks.

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- Maximize number of first choices.
- Minimize difference between preference ranks.
- Look for stable matchings

Stability.

Consider the couples:

- Alice and Bob
- Mary and John

Stability.

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- Mary and John

Bob prefers Mary to Alice.

Stability.

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Stability.

Consider the couples:

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Mary prefers Bob to John.

Uh...oh! Unstable pairing.

So..

Produce a pairing where there is no running off!

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Definition: A **pairing** is disjoint set of n man-woman pairs.

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Example: A pairing $S = \{(Bob, Alice); (John, Mary)\}$.

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Example: Bob and Mary are a rogue couple in S.

A stable pairing??

Given a set of preferences.

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Is there a stable pairing?
How does one find it?

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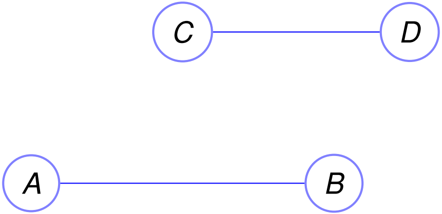
Consider a variant of this problem: stable roommates.

A	B	C	D
B	C	A	D
C	A	B	D
D	A	B	C



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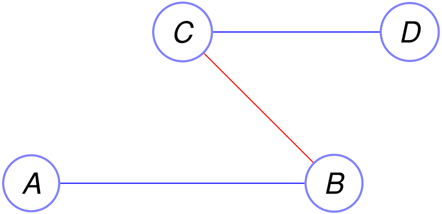

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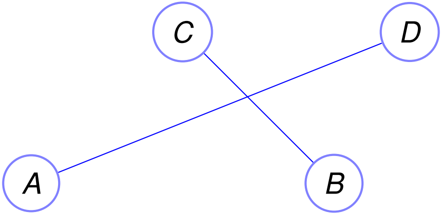

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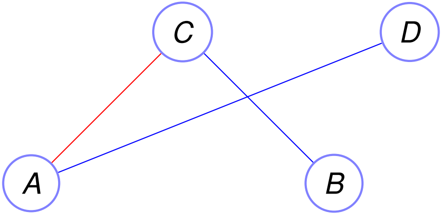

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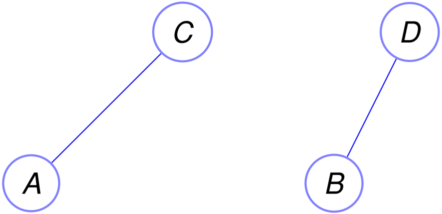

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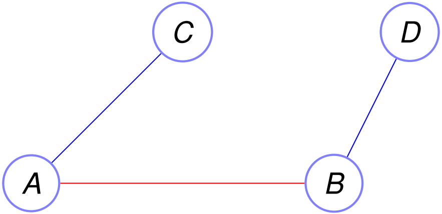

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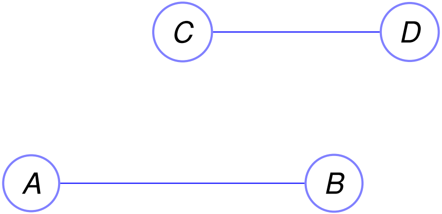

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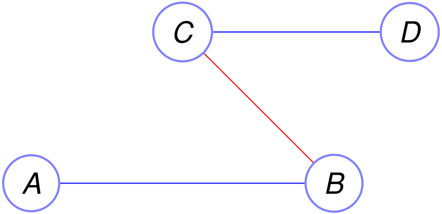

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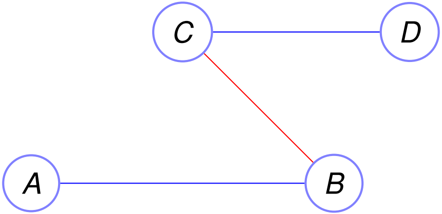

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1. Each man **proposes** to his favorite woman on his list.

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3. Rejected man crosses rejecting woman off his list.

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Stop when each woman gets exactly one proposal.

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Does this terminate?

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Do men or women do “better”?

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Stop when each woman gets exactly one proposal.
Does this terminate?

...produce a pairing?

....a stable pairing?

Do men or women do “better”?

Example.

Men			
A	1	2	3
B	1	2	3
C	2	1	3

Women			
1	C	A	B
2	A	B	C
3	A	C	B

Example.

Men			
A	1	2	3
B	1	2	3
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Women			
1	C	A	B
2	A	B	C
3	A	C	B

	Day 1	Day 2	Day 3	Day 4	Day 5
1					
2					
3					

Example.

Men			
A	1	2	3
B	1	2	3
C	2	1	3

Women			
1	C	A	B
2	A	B	C
3	A	C	B

	Day 1	Day 2	Day 3	Day 4	Day 5
1	A, B				
2	C				
3					

Example.

Men			
A	1	2	3
B	~~1~~	2	3
C	2	1	3

Women			
1	C	A	B
2	A	B	C
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	Day 1	Day 2	Day 3	Day 4	Day 5
1	A, ~~B~~				
2	C				
3					

Example.

Men			
A	1	2	3
B	~~1~~	2	3
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1	C	A	B
2	A	B	C
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	Day 1	Day 2	Day 3	Day 4	Day 5
1	A, ~~B~~	A			
2	C	B, C			
3					

Example.

Men			
A	1	2	3
B	~~1~~	2	3
C	~~2~~	1	3

Women			
1	C	A	B
2	A	B	C
3	A	C	B

	Day 1	Day 2	Day 3	Day 4	Day 5
1	A, ~~B~~	A			
2	C	B, ~~C~~			
3					

Example.

Men			
A	1	2	3
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Women			
1	C	A	B
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	Day 1	Day 2	Day 3	Day 4	Day 5
1	A, ~~B~~	A	A , C		
2	C	B, ~~C~~	B		
3					

Example.

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3					

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1	C	A	B
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	Day 1	Day 2	Day 3	Day 4	Day 5
1	A, ~~B~~	A	~~A~~ , C	C	
2	C	B, ~~C~~	B	A,B	
3					

Example.

Men			
A	~~1~~	2	3
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C	~~2~~	1	3

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1	C	A	B
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1	A, ~~B~~	A	~~A~~, C	C	
2	C	B, ~~C~~	B	A, ~~B~~	
3					

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A	~~1~~	2	3
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Example.

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1	C	A	B
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	Day 1	Day 2	Day 3	Day 4	Day 5
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2	C	B, ~~C~~	B	A, ~~B~~	A
3					B

Termination.

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Every non-terminated day a man **crossed** an item off the list.

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Total size of lists?

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Terminates in at most $n^2 + 1$ steps!

It gets better every day for women..

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Improvement Lemma:

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If man *b* proposes to a woman on day *k*,

It gets better every day for women..

Improvement Lemma:
If man b proposes to a woman on day k, every future day, she has on a string a man b' she likes at least as much as b.

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Proof:

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Proof:
Ind. Hyp.: $P(j)$ ($j \geq k$) — “Woman has as good an option on day j as on day k.”

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Base Case: $P(k)$: either she has no one/worse on a string (so puts b or better on a string), or she has someone better already.

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On day $j+1$, man $\hat{b}$ will come back (and possibly others).

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$\implies P(j+1)$. $\square$

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Contradiction.

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b^* ——— g^*

b ——— g

□

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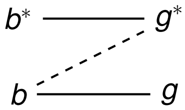

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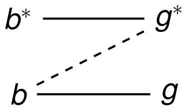

b likes g^* more than g.

g^* likes b more than b^*.

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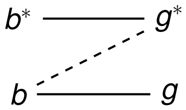

b likes g^* more than *g*.

g^* likes *b* more than b^*.

Man *b* proposes to g^* before proposing to *g*.

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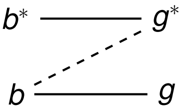

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Lemma: There is no rogue couple for the pairing formed by stable marriage algorithm.

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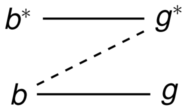

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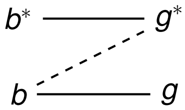

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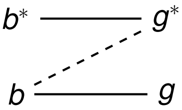

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Used Well-Ordering principle...

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Residency Matching..

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The method was used to match residents to hospitals.
Hospital optimal....
..until 1990's...Resident optimal.
Variations: couples!

Fun stuff from the Fall 2014 offering...

Follow the link.

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