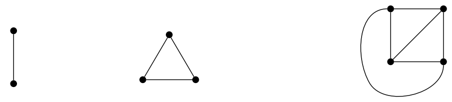
Today.

Types of graphs.

Complete Graphs.
Trees.
Hypercubes.

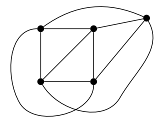
Complete Graph.



K_n complete graph on n vertices.
All edges are present.
Everyone is my neighbor.
Each vertex is adjacent to every other vertex.

How many edges?
Each vertex is incident to $n-1$ edges.
Sum of degrees is $n(n-1)$.
$\implies$ Number of edges is $n(n-1)/2$.
Remember sum of degree is $2|E|$.

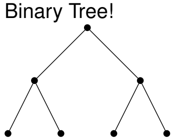
K_4 and K_5



K_5 is not planar.
Cannot be drawn in the plane without an edge crossing!
Prove it! Read Note 5!!

Trees!

Graph $G = (V, E)$.
Binary Tree!



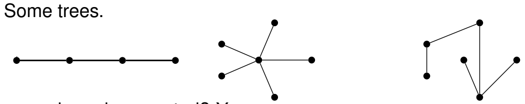
More generally.

Trees: Definitions

Definitions:

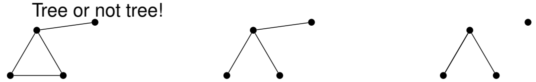
A connected graph without a cycle.
A connected graph with $|V|-1$ edges.
A connected graph where any edge removal disconnects it.
A connected graph where any edge addition creates a cycle.

Some trees.



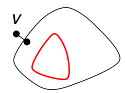
no cycle and connected? Yes.
$|V|-1$ edges and connected? Yes.
removing any edge disconnects it. Harder to check. but yes.
Adding any edge creates cycle. Harder to check. but yes.

Tree or not tree!



Equivalence of Definitions

Thm:
"G connected and has $|V|-1$ edges" $\equiv$
"G is connected and has no cycles."



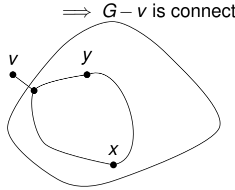
Proof of $\implies$ (only if): By induction on $|V|$.
Base Case: $|V| = 1$. $0 = |V|-1$ edges and has no cycles.

Induction Step: Assume for G with up to k vertices. Prove for $k+1$
Consider some vertex v in G. How is it connected to the rest of G?
Might it be connected by just 1 edge?
Is there a Degree 1 vertex?
Is the rest of G connected?

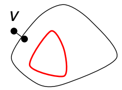
Equivalence of Definitions: Useful Lemma

Theorem:
"G connected and has $|V|-1$ edges" $\equiv$
"G is connected and has no cycles."

Lemma: If v is a degree 1 in connected graph G, $G-v$ is connected.
Proof:
For $x \neq v, y \neq v \in V$,
there is path between x and y in G since connected.
and does not use v (degree 1)
$\implies G-v$ is connected. □



Proof of only if.



Thm:
"G connected and has $|V|-1$ edges" $\equiv$
"G is connected and has no cycles."

Proof of $\implies$: By induction on $|V|$.
Base Case: $|V| = 1$. $0 = |V|-1$ edges and has no cycles.
Induction Step: Assume for G with up to k vertices. Prove for $k+1$
Claim: There is a degree 1 node.
Proof: First, connected $\implies$ every vertex degree ≥ 1.
Sum of degrees is $2|V|-2$
Average degree $2-(2/|V|)$
Not everyone is bigger than average! □
By degree 1 removal lemma, $G-v$ is connected.
$G-v$ has $|V|-1$ vertices and $|V|-2$ edges so by induction
$\implies$ no cycle in $G-v$.
And no cycle in G since degree 1 cannot participate in cycle. □

Proof of "if part"

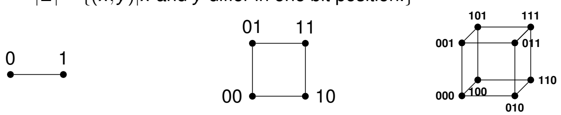
Thm:
"G is connected and has no cycles" $\implies$ "G connected and has $|V|-1$ edges"
Proof: Can we use the "degree 1" idea again?
Walk from a vertex using untraversed edges and vertices.
Until get stuck. Why? Finitely-many vertices, no cycle!
Claim: Degree 1 vertex.
Proof of Claim:
Can't visit more than once since no cycle.
Entered. Didn't leave. Only one incident edge. □
Removing node doesn't create cycle.
New graph is connected. (from our Degree 1 lemma).
By induction $G-v$ has $|V|-2$ edges.
G has one more or $|V|-1$ edges. □

Hypercubes.

Complete graphs, really well connected! Lots of edges.
$|V|(|V|-1)/2$
Trees, connected, few edges.
$(|V|-1)$

Hypercubes. Well connected. $|V|\log|V|$ edges!
Also represents bit-strings nicely.

$G = (V, E)$
$|V| = \{0,1\}^n$,
$|E| = \{(x,y) | x \text{ and } y \text{ differ in one bit position.}\}$

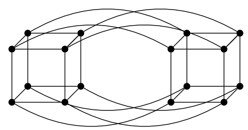


2^n vertices. number of n-bit strings!
$n2^{n-1}$ edges.
2^n vertices each of degree n
total degree is $n2^n$ and half as many edges!

Recursive Definition.

A 0-dimensional hypercube is a node labelled with the empty string of bits.

An n-dimensional hypercube consists of a 0-subcube (1-subcube) which is a $n-1$-dimensional hypercube with nodes labelled $0x$ ($1x$) with the additional edges $(0x, 1x)$.



Hypercube: Can't cut me!

Thm: Any subset S of the hypercube where $|S| \leq |V|/2$ has $\geq |S|$ edges connecting it to $V-S$: $|E \cap S \times (V-S)| \geq |S|$

Terminology:
$(S, V-S)$ is cut.
$(E \cap S \times (V-S))$ - cut edges.

Restatement: for any cut in the hypercube, the number of cut edges is at least the size of the small side.

Proof of Large Cuts.

Thm: For any cut $(S, V-S)$ in the hypercube, the number of cut edges is at least the size of the small side.
Proof:
Base Case: $n = 1$ V= $\{0,1\}$.
$S = \{0\}$ has one edge leaving.
$S = \emptyset$ has 0.

Induction Step Idea

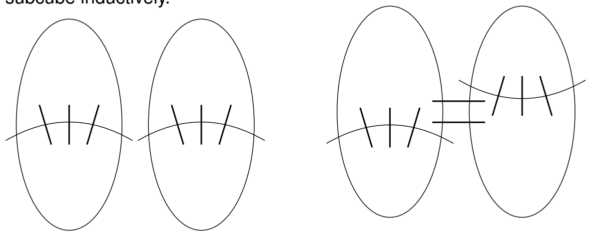
Thm: For any cut $(S, V-S)$ in the hypercube, the number of cut edges is at least the size of the small side.

Use recursive definition into two subcubes.

Two cubes connected by edges.

Case 1: Count edges inside subcube inductively.

Case 2: Count inside and across.



Induction Step

Thm: For any cut $(S, V-S)$ in the hypercube, the number of cut edges is at least the size of the small side, $|S|$.

Proof: Induction Step.
Recursive definition:
$H_0 = (V_0, E_0), H_1 = (V_1, E_1)$, edges E_x that connect them.
$H = (V_0 \cup V_1, E_0 \cup E_1 \cup E_x)$
$S = S_0 \cup S_1$ where S_0 in first, and S_1 in other.

Case 1: $|S_0| \leq |V_0|/2, |S_1| \leq |V_1|/2$
Both S_0 and S_1 are small sides. So by induction.
Edges cut in $H_0 \geq |S_0|$.
Edges cut in $H_1 \geq |S_1|$.

Total cut edges $\geq |S_0| + |S_1| = |S|$. □

Induction Step. Case 2.

Thm: For any cut $(S, V-S)$ in the hypercube, the number of cut edges is at least the size of the small side, $|S|$.
Proof: Induction Step. Case 2. $|S_0| \geq |V_0|/2$.
Recall Case 1: $|S_0|, |S_1| \leq |V|/2$
$|S_1| \leq |V_1|/2$ since $|S| \leq |V|/2$.
$\implies \geq |S_1|$ edges cut in E_1.
$|S_0| \geq |V_0|/2 \implies |V_0 - S_0| \leq |V_0|/2$
$\implies \geq |V_0| - |S_0|$ edges cut in E_0.

Edges in E_x connect corresponding nodes.
$\implies = |S_0| - |S_1|$ edges cut in E_x.

Total edges cut:
$\geq |S_1| + |V_0| - |S_0| + |S_0| - |S_1| = |V_0|$
$|V_0| = |V|/2 \geq |S|$. □
Also, case 3 where $|S_1| \geq |V|/2$ is symmetric.

Hypercubes and Boolean Functions.

The cuts in the hypercubes are exactly the transitions from 0 sets to 1 set on boolean functions on $\{0,1\}^n$.

Central area of study in computer science!

Yes/No Computer Programs $\equiv$ Boolean function on $\{0,1\}^n$

Central object of study.