Today.

Types of graphs.

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- Complete Graphs.
- Trees.
- Hypercubes.

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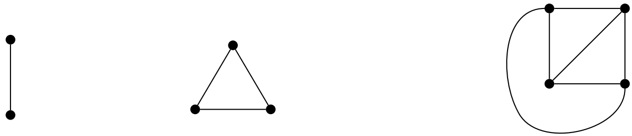
Types of graphs.

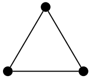
- Complete Graphs.
- Trees.
- Hypercubes.

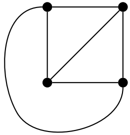
Complete Graph.



K_n complete graph on n vertices.

Complete Graph.

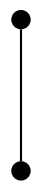


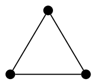


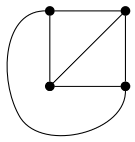
K_n complete graph on n vertices.

All edges are present.

Complete Graph.





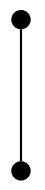


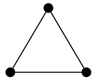
K_n complete graph on n vertices.

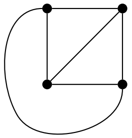
All edges are present.

Everyone is my neighbor.

Complete Graph.







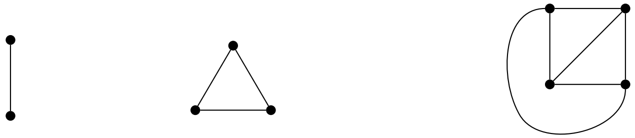
K_n complete graph on n vertices.

All edges are present.

Everyone is my neighbor.

Each vertex is adjacent to every other vertex.

Complete Graph.



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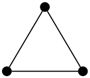
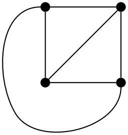
  

K_n complete graph on n vertices.

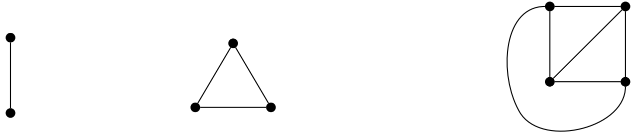
All edges are present.

Everyone is my neighbor.

Each vertex is adjacent to every other vertex.

How many edges?

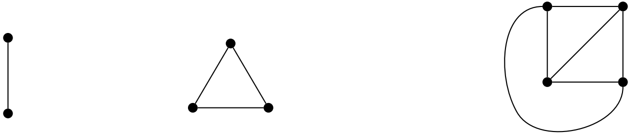
Complete Graph.



K_n complete graph on n vertices.
  All edges are present.
  Everyone is my neighbor.
  Each vertex is adjacent to every other vertex.

How many edges?
  Each vertex is incident to $n-1$ edges.

Complete Graph.



K_n complete graph on n vertices.

All edges are present.

Everyone is my neighbor.

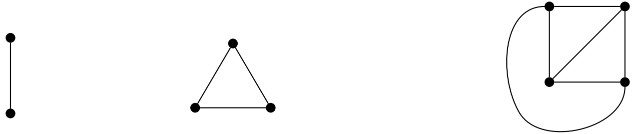
Each vertex is adjacent to every other vertex.

How many edges?

Each vertex is incident to $n-1$ edges.

Sum of degrees is $n(n-1)$.

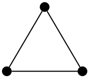
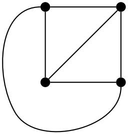
Complete Graph.



K_n complete graph on n vertices.
  All edges are present.
  Everyone is my neighbor.
  Each vertex is adjacent to every other vertex.

How many edges?
  Each vertex is incident to $n-1$ edges.
  Sum of degrees is $n(n-1)$.
  $\implies$ Number of edges is $n(n-1)/2$.

Complete Graph.

K_n complete graph on n vertices.

All edges are present.

Everyone is my neighbor.

Each vertex is adjacent to every other vertex.

How many edges?

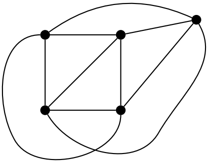
Each vertex is incident to $n-1$ edges.

Sum of degrees is $n(n-1)$.

$\implies$ Number of edges is $n(n-1)/2$.

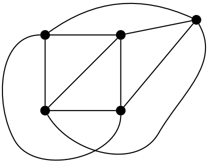
Remember sum of degree is $2|E|$.

K_4 and K_5



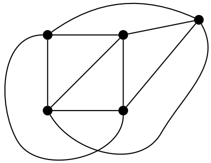
K_5 is not planar.

K_4 and K_5



K_5 is not planar.
Cannot be drawn in the plane without an edge crossing!

K_4 and K_5

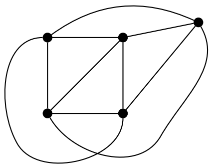


K_5 is not planar.

Cannot be drawn in the plane without an edge crossing!

Prove it!

K_4 and K_5



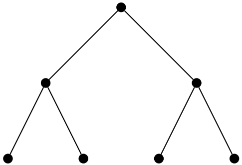
K_5 is not planar.

Cannot be drawn in the plane without an edge crossing!

Prove it! Read Note 5!!

Trees!

Graph $G = (V, E)$.
Binary Tree!



More generally.

Trees: Definitions

Definitions:

Trees: Definitions

Definitions:

A connected graph without a cycle.

Trees: Definitions

Definitions:

A connected graph without a cycle.
A connected graph with $|V| - 1$ edges.

Trees: Definitions

Definitions:

A connected graph without a cycle.
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A connected graph where any edge removal disconnects it.

Trees: Definitions

Definitions:

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Trees: Definitions

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Trees: Definitions

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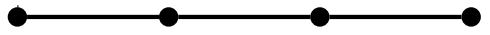
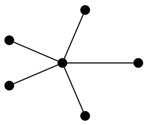
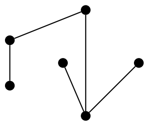
Some trees.







no cycle and connected?

Trees: Definitions

Definitions:

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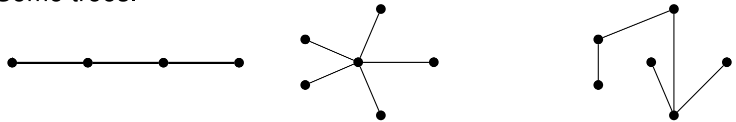
Some trees.



no cycle and connected? Yes.

Trees: Definitions

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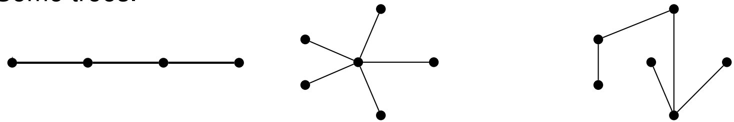
Some trees.



no cycle and connected? Yes.
$|V|-1$ edges and connected?

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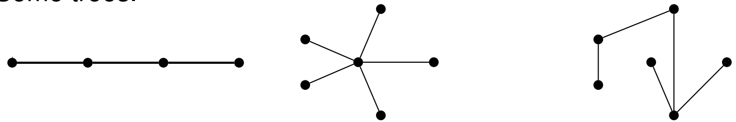

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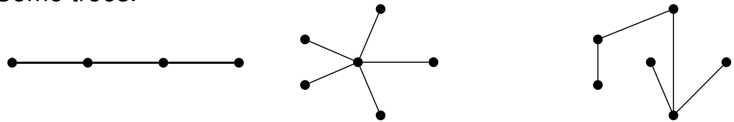

no cycle and connected? Yes.
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removing any edge disconnects it.

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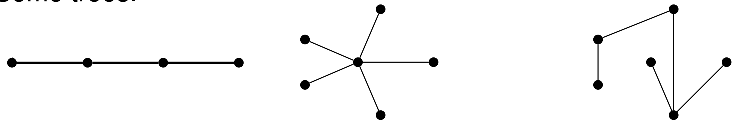

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$|V|-1$ edges and connected? Yes.
removing any edge disconnects it. Harder to check.

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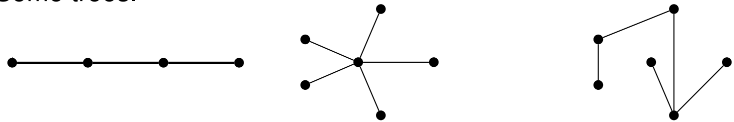

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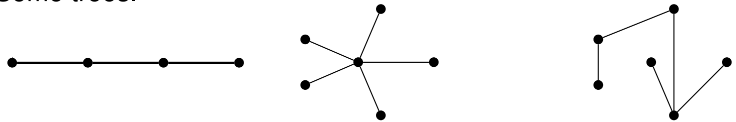

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Adding any edge creates cycle.

Trees: Definitions

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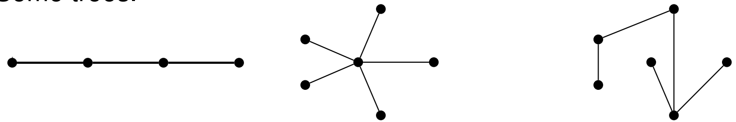

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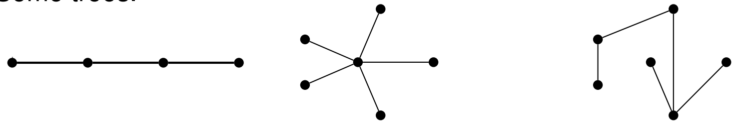

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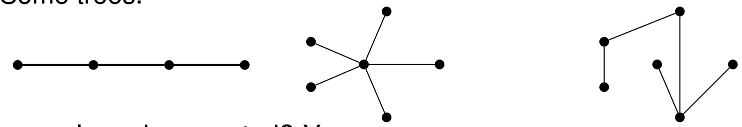

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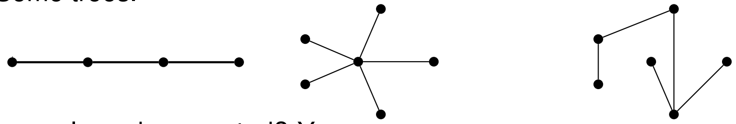

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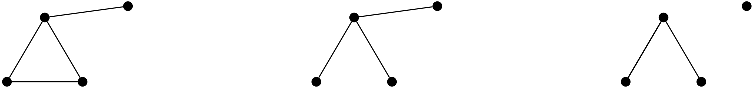
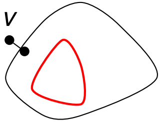
$|V| - 1$ edges and connected? Yes.
removing any edge disconnects it. Harder to check. but yes.
Adding any edge creates cycle. Harder to check. but yes.

Tree or not tree!



Equivalence of Definitions

Thm:
"G connected and has $|V| - 1$ edges" $\equiv$
"G is connected and has no cycles."



Equivalence of Definitions

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"G connected and has $|V|-1$ edges" $\equiv$
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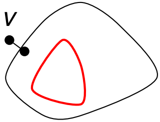
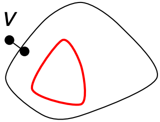


Proof of $\implies$ (only if):

Equivalence of Definitions



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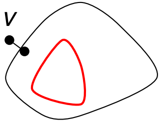
Proof of $\implies$ (only if): By induction on $|V|$.

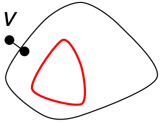
Equivalence of Definitions



Thm:
"G connected and has $|V|-1$ edges" $\equiv$
"G is connected and has no cycles."

Proof of $\implies$ (only if): By induction on $|V|$.
Base Case: $|V|=1$. $0=|V|-1$ edges and has no cycles.

Equivalence of Definitions



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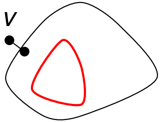
Proof of $\implies$ (only if): By induction on $|V|$.
Base Case: $|V| = 1$. $0 = |V| - 1$ edges and has no cycles.

Equivalence of Definitions



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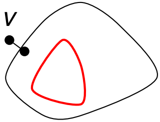
Proof of $\implies$ (only if): By induction on $|V|$.
Base Case: $|V|=1$. $0=|V|-1$ edges and has no cycles.

Induction Step: Assume for G with up to k vertices. Prove for $k+1$

Equivalence of Definitions



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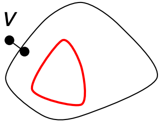
Proof of $\implies$ (only if): By induction on $|V|$.
Base Case: $|V|=1$. $0=|V|-1$ edges and has no cycles.

Induction Step: Assume for G with up to k vertices. Prove for $k+1$
Consider some vertex v in G. How is it connected to the rest of G?

Equivalence of Definitions



Thm:
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Proof of $\Longrightarrow$ (only if): By induction on $|V|$.
Base Case: $|V|=1$. $0=|V|-1$ edges and has no cycles.

Induction Step: Assume for G with up to k vertices. Prove for $k+1$
Consider some vertex v in G. How is it connected to the rest of G?
Might it be connected by just 1 edge?

Equivalence of Definitions

Thm:
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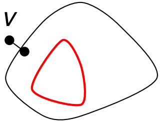

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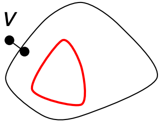
Induction Step: Assume for G with up to k vertices. Prove for $k+1$
Consider some vertex v in G. How is it connected to the rest of G?
Might it be connected by just 1 edge?
Is there a Degree 1 vertex?

Equivalence of Definitions



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Proof of $\implies$ (only if): By induction on $|V|$.
Base Case: $|V| = 1$. $0 = |V| - 1$ edges and has no cycles.

Induction Step: Assume for G with up to k vertices. Prove for $k+1$
Consider some vertex v in G. How is it connected to the rest of G?
Might it be connected by just 1 edge?
Is there a Degree 1 vertex?
Is the rest of G connected?

Equivalence of Definitions: Useful Lemma

Theorem:
"G connected and has $|V|-1$ edges" $\equiv$
"G is connected and has no cycles."

Lemma: If v is a degree 1 in connected graph G, $G-v$ is connected.
Proof:
For $x \neq v, y \neq v \in V$,

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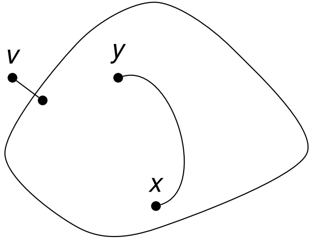

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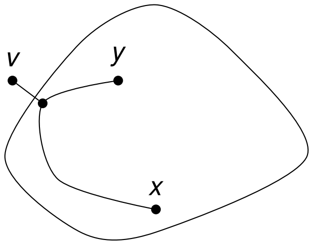

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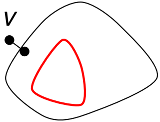

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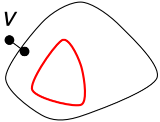

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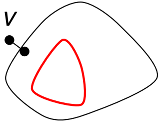

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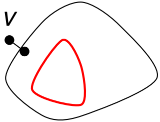

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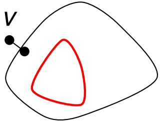
Sum of degrees is $2|V|-2$

Proof of only if.



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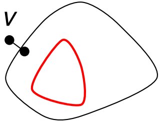

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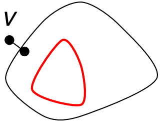

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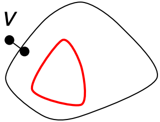

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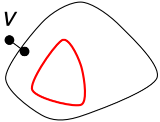

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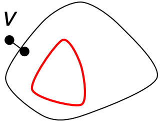

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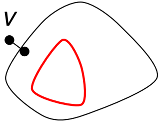

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Proof of “if part”

Thm:
“G is connected and has no cycles” $\Longrightarrow$ “G connected and has $|V|-1$ edges”
Proof:

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Walk from a vertex using untraversed edges and vertices.

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Until get stuck. Why?

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Proof of Claim:
Can't visit more than once since no cycle.

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Hypercubes.

Complete graphs, really well connected!

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$|V|(|V|-1)/2$

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$|V|(|V|-1)/2$

Trees, connected, few edges.

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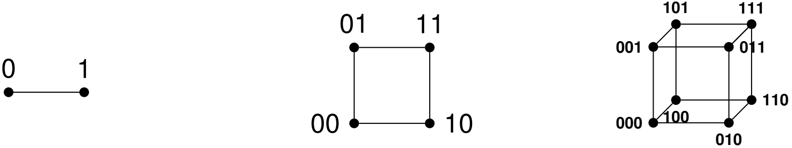

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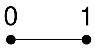

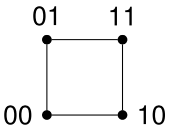

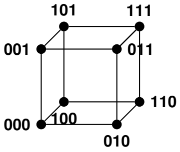

2^n vertices.

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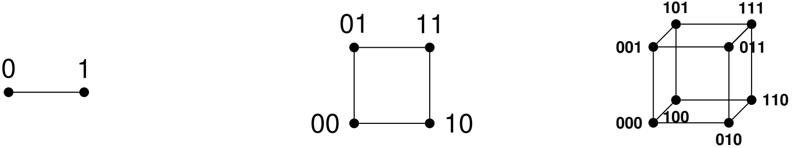

2^n vertices. number of n-bit strings!

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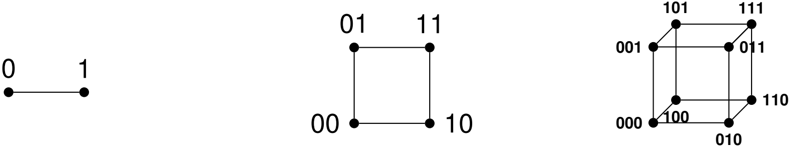

2^n vertices. number of n-bit strings!

$n2^{n-1}$ edges.

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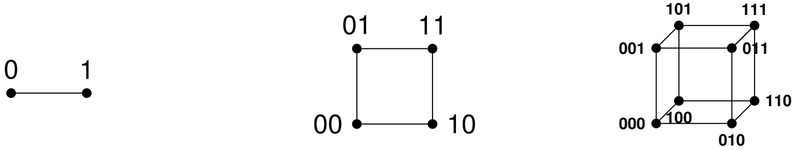

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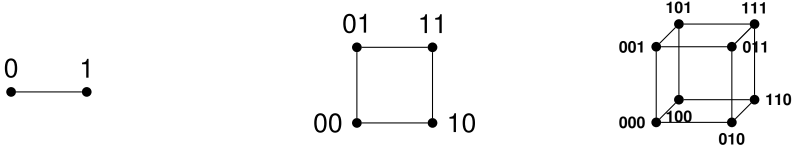

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total degree is $n2^n$

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Hypercubes. Well connected. $|V|\log|V|$ edges!

Also represents bit-strings nicely.

$G=(V,E)$

$|V|=\{0,1\}^n$,

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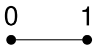

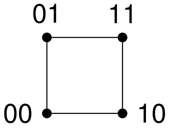

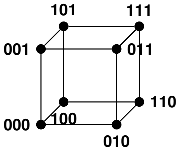

2^n vertices. number of n-bit strings!

$n2^{n-1}$ edges.

2^n vertices each of degree n

total degree is $n2^n$ and half as many edges!

Hypercubes.

Complete graphs, really well connected! Lots of edges.

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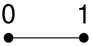

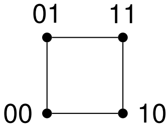

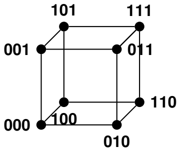

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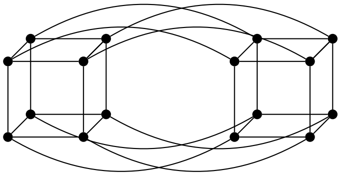

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Restatement: for any cut in the hypercube, the number of cut edges is at least the size of the small side.

Proof of Large Cuts.

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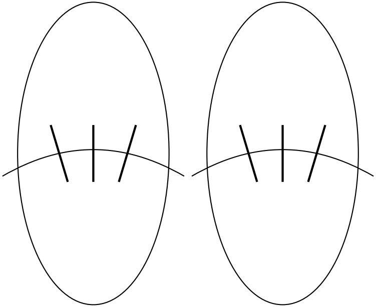

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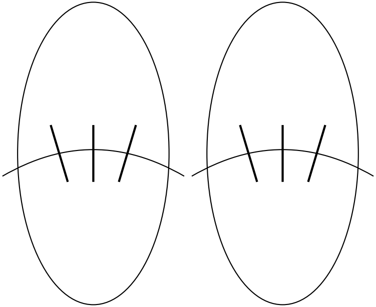

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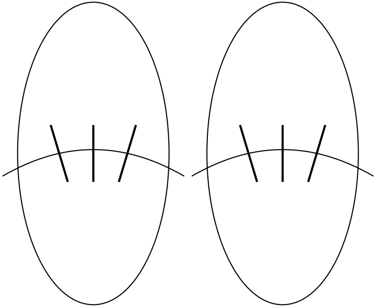

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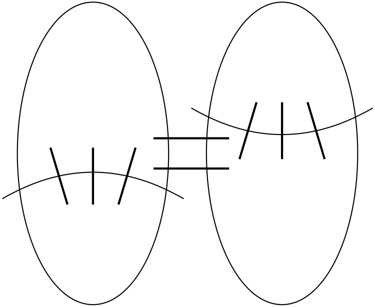

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Induction Step. Case 2.

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Proof: Induction Step. Case 2. $|S_0| \geq |V_0|/2$.

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Edges in E_x connect corresponding nodes.

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Induction Step. Case 2.

Thm: For any cut $(S, V - S)$ in the hypercube, the number of cut edges is at least the size of the small side, $|S|$.
Proof: Induction Step. Case 2. $|S_0| \geq |V_0|/2$.
Recall Case 1: $|S_0|, |S_1| \leq |V|/2$
$|S_1| \leq |V_1|/2$ since $|S| \leq |V|/2$.
$\implies \geq |S_1|$ edges cut in E_1.
$|S_0| \geq |V_0|/2 \implies |V_0 - S_0| \leq |V_0|/2$
$\implies \geq |V_0| - |S_0|$ edges cut in E_0.

Edges in E_x connect corresponding nodes.
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Also, case 3 where $|S_1| \geq |V|/2$ is symmetric.

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