Midterm 2 Review.

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Midterm Topics: Notes 6-14.

Modular Arithmetic. Inverses. GCD/Extended-GCD.

RSA/Cryptography.

Polynomials.
  Secret Sharing.
  Erasure Resistant Encoding.
  Error Correction.

Counting.

Countability.

Computability.

Probability Topics covered by Prof. Walrand.

Midterm format

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Will broadly follow Midterm1 format:
mix of short and longer questions

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Be familiar with Midterm1 topics... but MT2 will focus on Notes 6-14.

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Property 1 gives uniqueness.

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Recover Message: Any n packets are sufficient by property 2.

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Property 2: There is exactly 1 polynomial of degree $\leq d$ with arithmetic modulo prime p that contains any $d+1$: $(x_1, y_1), \ldots, (x_{d+1}, y_{d+1})$ with x_i distinct.

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Idea: Error locator polynomial of degree k with zeros at errors.

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Isomorphism principle.

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Given a function, $f : D \to R$.

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$[0,1]$ is same cardinality as nonnegative reals!

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All countably infinite sets are the same cardinality as each other.

Examples

Countably infinite (same cardinality as naturals)

- E even numbers.

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  Where are the odds?

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(The set of all subsets of S, is the **powerset** of N.)

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2. Otherwise, halt immediately.

Assumption: there is a program HALT.
There is text that “is” the program HALT.
There is text that is the program Turing.
Can run Turing on Turing!

Does Turing(Turing) halt?

Turing(Turing) halts
$\implies$ then HALTS(Turing, Turing) $=$ halts
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Halt and Turing.

Theorem: There is no program HALT.
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$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

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Counting

First Rule

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First Rule
Second Rule

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Stars/Bars

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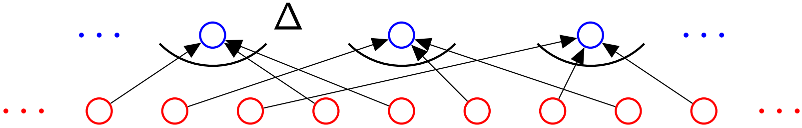
Example: visualize.

First rule: $n_1 \times n_2 \cdots \times n_3$. **Product Rule.**

Second rule: when order doesn't matter divide..when possible.



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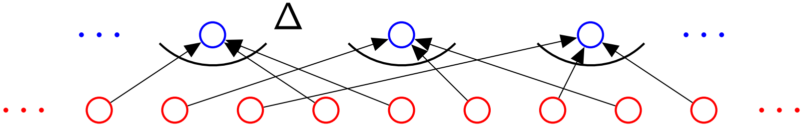

3 card Poker deals: 52

Example: visualize.

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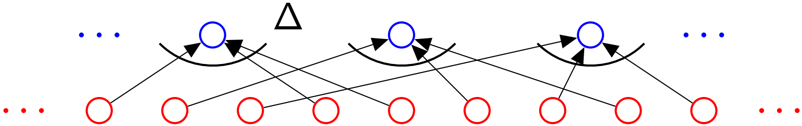

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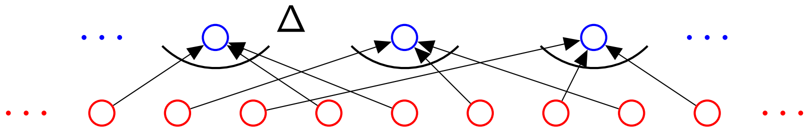

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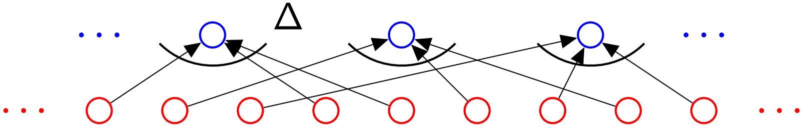

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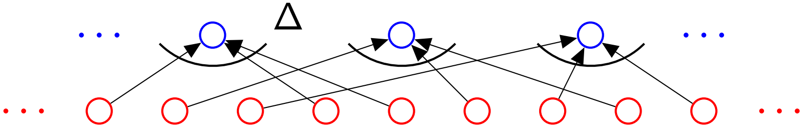

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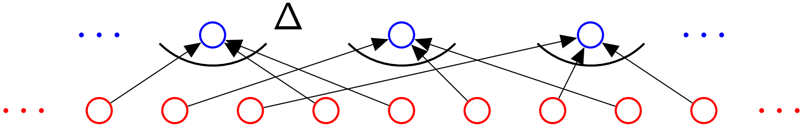

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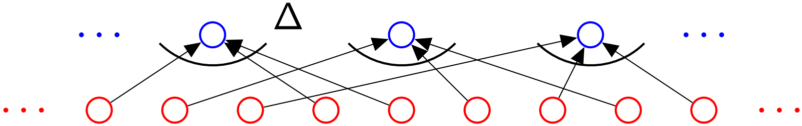

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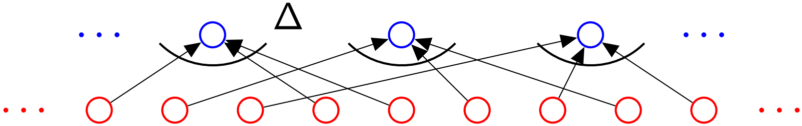

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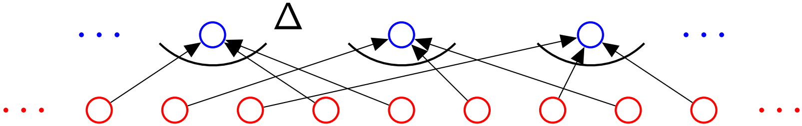

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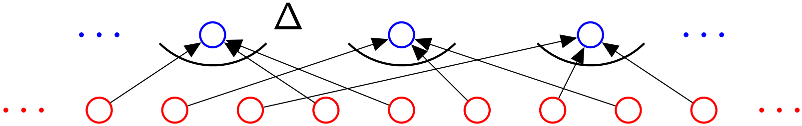

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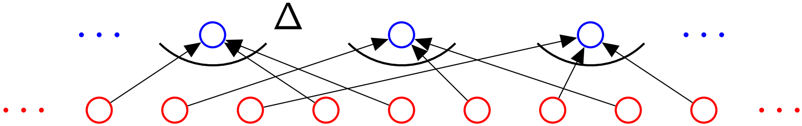

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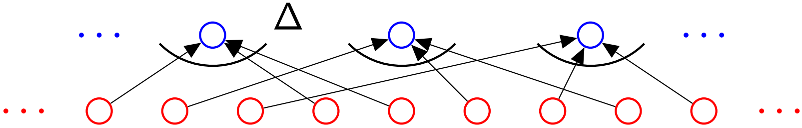

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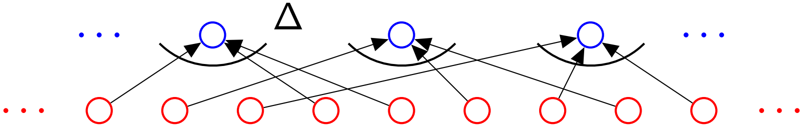

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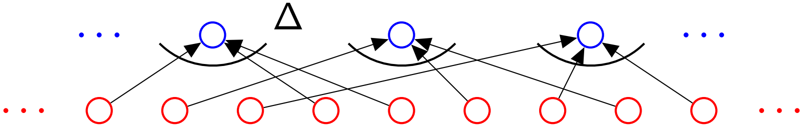

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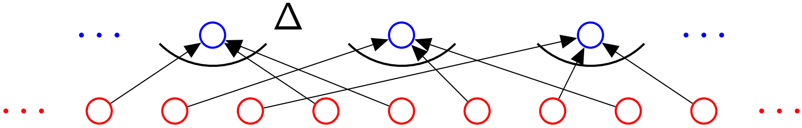

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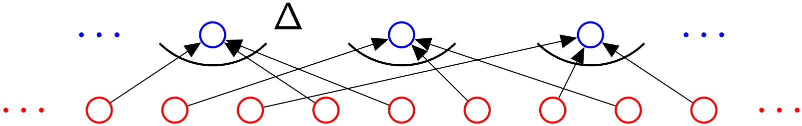

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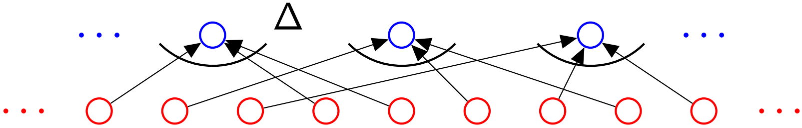

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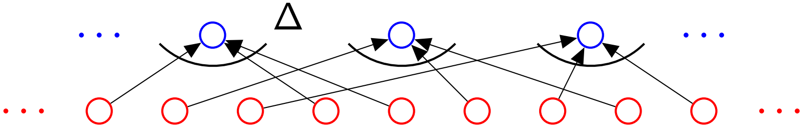

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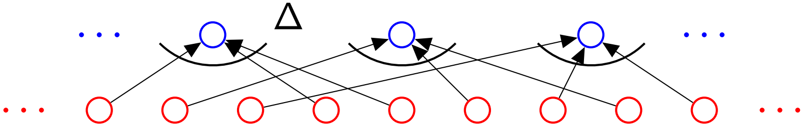

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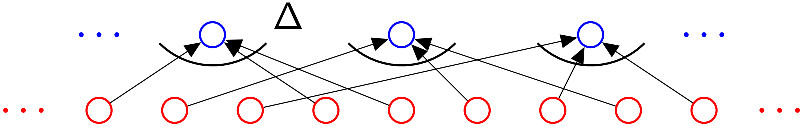
Deals: Q, K, A, Q, A, K, K, A, Q, K, A, Q, A, K, Q, A, Q, K.
$\Delta = 3 \times 2 \times 1$ First rule again.
Total: $\frac{52!}{49!3!}$ Second Rule!

Choose k out of n.
Ordered set: $\frac{n!}{(n-k)!}$
What is Δ? $k!$

Example: visualize.

First rule: $n_1 \times n_2 \cdots \times n_3$**. Product Rule.**
Second rule: when order doesn't matter divide..when possible.



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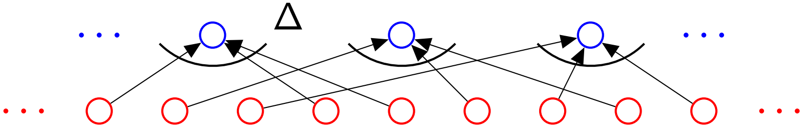

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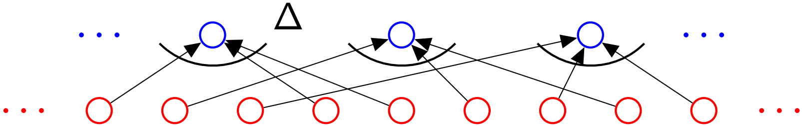

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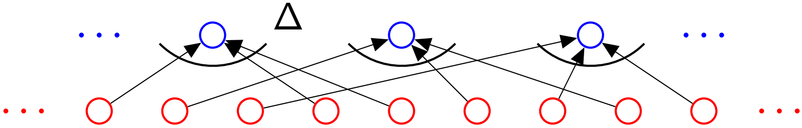

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Answer: $|S| + |T| - |S \cap T| = 10^9 + 10^9 - 10^8$.

Combinatorial Proofs.

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Good Studying and Good Luck!!!