Stable Marriage Problem

Introduced by Gale and Shapley in a 1962 paper in the American Mathematical Monthly.

Proved useful in many settings, led eventually to 2012 Nobel Prize in Economics (to Shapley and Roth).

Original Problem Setting:

- Small town with n men and n women.
- Each woman has a ranked preference list of men.
- Each man has a ranked preference list of women.

How should they be matched?

What criteria to use?

- Maximize number of first choices.
- Minimize difference between preference ranks.
- Look for stable matchings

Stability.

Consider the couples:

- Alice and Bob
- Mary and John

Bob prefers Mary to Alice.

Mary prefers Bob to John.

Uh...oh! Unstable pairing.

So..

Produce a pairing where there is no running off!

Definition: A **pairing** is disjoint set of n man-woman pairs.

Example: A pairing $S = \{(Bob, Alice); (John, Mary)\}$.

Definition: A **rogue couple** b, g for a pairing S:
b and g prefer each other to their partners in S

Example: Bob and Mary are a rogue couple in S.

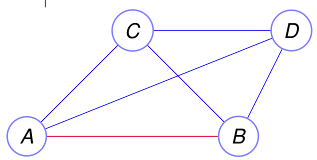
A stable pairing??

Given a set of preferences.

Is there a stable pairing?
How does one find it?

Consider a variant of this problem: stable roommates.

A	B	C	D
B	C	A	D
C	A	B	D
D	A	B	C



The Stable Marriage Algorithm.

Each Day:

1. Each man **proposes** to his favorite woman on his list.
2. Each woman rejects all but her favorite proposer (whom she puts on a string.)
3. Rejected man crosses rejecting woman off his list.

Stop when each woman gets exactly one proposal.
Does this terminate?

...produce a pairing?

....a stable pairing?

Do men or women do "better"?

Example.

Men			
A	~~1~~	2	3
B	~~1~~	~~2~~	3
C	~~2~~	1	3

Women			
1	C	A	B
2	A	B	C
3	A	C	B

	Day 1	Day 2	Day 3	Day 4	Day 5
1	A, ~~B~~	A	~~A~~, C	C	C
2	C	B, ~~C~~	B	A, ~~B~~	A
3					B

Termination.

Every non-terminated day a man **crossed** an item off the list.

Total size of lists? n men, n length list. n^2

Terminates in at most $n^2 + 1$ steps!

It gets better every day for women..

Improvement Lemma:
If man b proposes to a woman on day k, every future day, she has on a string a man b' she likes at least as much as b.
(that is, her options get better)

Proof:
Ind. Hyp.: $P(j)$ ($j \geq k$) — "Woman has as good an option on day j as on day k."

Base Case: $P(k)$: either she has no one/worse on a string (so puts b or better on a string), or she has someone better already.

Assume $P(j)$. Let $\hat{b}$ be man on string on day $j \geq k$. So $\hat{b}$ is as good as b.

On day $j+1$, man $\hat{b}$ will come back (and possibly others).

Woman can choose $\hat{b}$ just as well, or pick a better option.

$\implies P(j+1)$. □

Pairing when done.

Lemma: Every man is matched at end.

Proof:
If not, a man b must have been rejected n times.

Every woman has been proposed to by b,
and Improvement lemma

$\implies$ each woman has a man on a string.

and each man on at most one string.

n women and n men. Same number of each.

$\implies$ b must be on some woman's string!

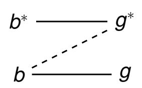
Contradiction.

□

Pairing is Stable.

Lemma: There is no rogue couple for the pairing formed by stable marriage algorithm.

Proof:
Assume there is a rogue couple; (b, g^*)



b likes g^* more than g.

g^* likes b more than b^*.

Man b proposes to g^* before proposing to g.

So g^* rejected b (since he moved on)

By improvement lemma, g^* likes b^* better than b.

Contradiction! □

Good for men? women?

Is the SMA better for men? for women?

Definition: A **pairing is** x**-optimal** if x's partner
is its best partner in any stable pairing.

Definition: A **pairing is** x**-pessimal** if x's partner
is its worst partner in any stable pairing.

Definition: A **pairing is man optimal** if it is x-optimal for **all** men x.

..and so on for man pessimal, woman optimal, woman pessimal.

Claim: The optimal partner for a man must be first in his preference list.

True? False? False!

Subtlety here: Best partner in any stable pairing.
As well as you can in a globally stable solution!

Question: Is there a even man or woman optimal pairing?

SMA is optimal!

For men? For women?

Theorem: SMA produces a man-optimal pairing.

Proof:
Assume not: there are men who do not get their optimal woman.

Let t be first day *any* man b gets rejected
by his optimal woman g who he is paired with
in some stable pairing S.

Let g put b^* on a string in place of b on day $t \implies g$ prefers b^* to b

By choice of day t, b^* has not yet been rejected by his optimal woman.

Therefore, b^* prefers g to optimal woman, and hence to his partner g^* in S.

Rogue couple for S.
So S is not a stable pairing. Contradiction. □

Recap: S - stable. $(b^*, g^*) \in S$. But (b^*, g) is rogue couple!

Used Well-Ordering principle...

How about for women?

Theorem: SMA produces woman-pessimal pairing.

T – pairing produced by SMA.

S – worse stable pairing for woman g.

In T, (g, b) is pair.

In S, (g, b^*) is pair. b is paired with someone else, say g^*.

g likes b^* less than she likes b.

T is man optimal, so b likes g more than g^*, his partner in S.

(g, b) is Rogue couple for S

S is not stable.

Contradiction. □

Residency Matching..

The method was used to match residents to hospitals.
Hospital optimal....
..until 1990's...Resident optimal.
Variations: couples!

Fun stuff from the Fall 2014 offering...

Follow the link.

Link